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Linearized Solution of Isentropic Motions of a Perfect Fluid in General Relativity

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Isentropic motions of a perfect fluid are studied by using comoving coordinates in the framework of general relativity without assuming any symmetry in the line element and a linearized solution is obtained by dealing with the Cauchy problem.

1. INTRODUCTION

In view of the fact that the equations of general relativity are highly nonlinear, problems of relativistic astrophysics and cosmology are in general solved by assuming various symmetries in the line element. But there occur some important problems where one cannot assume any symmetry in the line element and consequently it poses an uphill task to solve such problems. With a view to get rid of this difficulty one may, of course, attack the problem by dealing with the Cauchy problem in the framework of general relativity and indeed some attempts have been made in this direction. Pachner (1968, 1971), Bera and Datta (1974, 1975), Datta (1975-76, 1976-77), and Basu et al. (to be published) have studied the isentropic motions of a perfect fluid by using comoving coordinates in the framework of general relativity without assuming any symmetry in the line element. The solutions obtained by these authors by dealing with the Cauchy problem for a perfect fluid correspond to linear approximation.

In the present paper we have, however, pursued the study further. A new solution is obtained on the assumption that all thermodynamic processes are adiabatic and that only mechanical energy may be released in the process. The solution arrived at is distinct from Pachner's (1971) and is more general than that of Bera and Datta (1974).

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2. NOTATION

The Greek indices run from 1 to 4, Latin indices from 1 to 3. Summation convention is followed throughout. The signature of the metric is +2. A system of units is used in which the velocity of light and the Newtonian constant of gravitation are each equal to unity. A comma followed by an index denotes ordinary partial differentiation, while a semicolon followed by an index denotes covariant derivative. We define g as $\det g_{\mu\nu}$, p the proper pressure, ρ the proper rest mass density, and ϵ the proper internal energy per unit mass. The Krönecker delta function is defined by

$$\delta_\mu^\nu = 1 \text{ or } 0 \text{ as } \mu = \nu \text{ or } \mu \neq \nu$$

The timelike 4-velocity is denoted by u^ν, where

$$u^\nu \equiv \frac{dx^\nu}{ds}$$

3. BASIC EQUATIONS OF THE PROBLEM

We consider the general form of the metric to be

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

where the $g_{\mu\nu}$'s are functions of x^1, x^2, x^3, and x^4. We introduce a system of comoving reference frames defined by

$$u^i = \delta_4^i \qquad (i = 1,2,3) \tag{2}$$

The timelike 4-velocity u^ν satisfies the identities

$$u^\nu u_\nu = -1 \tag{3}$$

which gives in view of (2)

$$u^4 = (-g_{44})^{-1/2} \tag{4}$$

The conservation law of the baryon number

$$(\rho u^\nu)_{;\nu} = 0 \tag{5}$$

gives in the case on integration with respect to time

$$\rho = (g_{44}/g)^{1/2} f(x^i) \tag{6}$$

where $f(x^i)$ is a function of space coordinates and may be determined by the initial distribution of matter. The equation (6) is called the equation of continuity, which connects the proper rest mass density, a measurable physical quantity with the components of the metric tensor.

The energy–momentum tensor $T_{\mu\nu}$ for a perfect fluid is defined by

$$T_\mu^\nu = \bar{\rho} u_\mu u^\nu + p\delta_\mu^\nu, \tag{7}$$

where $\bar{\rho}$, the enthalpy density, is given by

$$\bar{\rho} = \rho\mu \tag{8}$$

and

$$\mu = 1 + \epsilon + p/\rho \tag{9}$$

In phenomenological thermodynamics the enthalpy satisfies the relation

$$d\mu = dp/\rho + TdS \tag{10}$$

where the function S is defined as specific entropy and T, the temperature. It is clear from (10) that of the three state functions μ, p, and ρ, only two can be independent.

In view of (3), (5), and (7)–(10), the equations of motion for a perfect fluid,

$$T_{\mu;\nu}^\nu = 0 \tag{11}$$

reduce to (Datta, 1975-76, 1976-77; Krasiński, 1973, 1977; Plebański, 1970)

$$[(\mu u_\lambda)_{,\nu} - (\mu u_\nu)_{,\lambda}]u^\nu - TS_{,\lambda} = 0 \tag{12}$$

For isentropic motions of a perfect fluid we have

$$S_{,\lambda} = 0 \tag{13}$$

Thus in view of (2) and (13), equations (12) assume the form

$$(\mu u_\lambda)_{,4} - (\mu u_4)_{,\lambda} = 0 \tag{14}$$

and the enthalpy relation (10) reduces to

$$d\mu = dp/\rho \tag{15}$$

Thus for isentropic motions one has from (9) and (15)

$$\frac{d\epsilon}{d\rho} = p/\rho^2 \tag{16}$$

This equation also follows from the law of conservation of energy

$$T^{\nu}_{4;\,\nu} = 0 \tag{17}$$

by virtue of the equation of continuity (5) in the system of comoving reference frames defined by (2). The equation of continuity (5) is postulated in this case independently of equations (11). Equation (16) expresses the conservation of energy in an ideal fluid at a constant entropy. It is evident that of the four equations (14) the three equations

$$(\mu u_i)_{,4} - (\mu u_4)_{,i} = 0 \tag{18}$$

are independent and the fourth one is satisfied automatically. Accordingly, the four conservation laws of energy–momentum (11) are replaced by the four equations (16) and (18).

We consider the case where all thermodynamical processes are adiabatic and assume

$$\epsilon + p/\rho = k(x^4) \tag{19}$$

where $k(x^4)$ is a function of space coordinates. As will be evident from our later discussion this assumption seems to be interesting and useful from the point of view of mathematical simplicity.

The conservation law of energy (16) implies that

$$p = p(\rho)$$
$$\epsilon = \epsilon(\rho) \tag{20a}$$

or in other words

$$\rho = \rho(p)$$
$$\epsilon = \epsilon(p) \tag{20b}$$

The field equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=-8\pi T_{\mu\nu} \tag{21}$$

are equivalent, according to Synge (Synge, 1966; Lichnerowicz, 1955), to six independent Einstein field equations

$$R_{ij}=-8\pi\left(T_{ij}-\tfrac{1}{2}Tg_{ij}\right) \tag{22}$$

where

$$T=g^{\alpha\beta}T_{\alpha\beta} \tag{23}$$

and four laws of conservation of energy–momentum given by (16) and (18). Furthermore, one has to satisfy four consistency conditions

$$R_{4\nu}-\tfrac{1}{2}Rg_{4\nu}=-8\pi T_{4\nu}, \qquad g^{44}\neq 0 \tag{24}$$

on the hypersurface $x^4=0$.

Now the behavior of isentropic motions of a perfect fluid is described by the nine equations (18) and (22) together with the consistency conditions (24).

Next, we make a plausible assumption

$$g_{44}=1/\mu^2 \tag{25}$$

Then (4) and (6), respectively, reduce with the help of (25) to

$$u^4=\mu \tag{26}$$

$$\rho=f(x^i)/\mu(-g)^{1/2} \tag{27}$$

Now we have nine equations (18) and (22) to determine nine metric tensor components g_{ik} and g_{i4} together with consistency conditions (24).

In view of (2), (25), and (26), equations (18) yield on integration with respect to time

$$g_{i4}=C_i(x^j)/\mu^2 \tag{28}$$

where $C_i(x^j)$ are three functions of space coordinates and may be determined by the initial conditions.

Again in view of (2), (25), and (26) the covariant components u_ν are given by

$$u_\nu = C_\nu/\mu \tag{29}$$

with

$$C_4 = -1 \tag{30}$$

The functions C_ν which represent the covariant components of a 4-vector are known as Coriolis potentials (Tauber and Weinberg, 1961), which are connected with the metric tensor components $g_{\nu 4}$ by means of equations (28) and (29). A comoving reference system remains comoving if one applies the coordinate transformations of the type (Tauber and Weinberg, 1961)

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3), \qquad \bar{x}^4 = x^4 \tag{31}$$

If we assume that the vector field u_i and the proper rest mass density ρ vary sufficiently smoothly on the hypersurface $x^4 = 0$, we obtain by means of the field equations (Bera and Datta, 1974, 1975; Pachner, 1971).

$$C_1 = C_2 = 0, \qquad C_3 = C_3(x^2, x^3), \qquad C_4 = -1 \tag{32}$$

and by using the following coordinate transformations as permitted by (31)

$$\begin{aligned} &\bar{x}^1 = \bar{x}^1(x^1, x^2, x^3), \qquad &&\bar{x}^2 = \bar{x}^2(x^2, x^3) \\ &\bar{x}^3 = \bar{x}^3(x^3), &&\bar{x}^4 = x^4 \end{aligned} \tag{33}$$

Now in the system where the Coriolis potentials C_ν are given by (32) the equations (29) and (30) give

$$\begin{aligned} &u_1 = g_{14} = 0, \qquad &&u_2 = g_{24} = 0, \\ &u_3 = \mu g_{34} = C_3/\mu, &&u_4 = \mu g_{44} = -1/\mu \end{aligned} \tag{34}$$

The contravariant angular velocity vector is given by the formula (Gödel, 1949; Taub, 1956)

$$\Omega^\nu = \tfrac{1}{2}(-g)^{-1/2} e^{\nu\alpha\beta\gamma} u_\alpha u_{\beta,\gamma} \tag{35}$$

where the contravariant skew-symmetric tensor density $e^{\nu\alpha\beta\gamma}$ is defined by

$$e^{\nu\alpha\beta\gamma}=1,-1,0 \tag{36}$$

In terms of three-dimensional vector notations, the equations (35) reduce to

$$\Omega^{\nu}=(\mathbf{\Omega},\Omega^{4}) \tag{37}$$

where the three-dimensional vector $\mathbf{\Omega}$ is given, in our case, by

$$\mathbf{\Omega}=\tfrac{1}{2}(-g)^{-1/2}\operatorname{curl}\mathbf{C}/\mu^{2} \tag{38}$$

and

$$\Omega^{4}=0 \tag{39}$$

$\mathbf{C}$ being a three-dimensional vector with components C_i. Now $\mathbf{\Omega}$ has only one nonvanishing component

$$\Omega^{1}=\tfrac{1}{2}(-g)^{-1/2}C_{3,2}/\mu^{2}, \qquad \Omega^{2}=\Omega^{3}=0 \tag{40}$$

and has the direction of the x^1 axis everywhere. Furthermore, it is evident from (32) and (35) that

$$\Omega^{\nu}C_{\nu}=0 \tag{41}$$

and

$$\Omega^{\nu}u_{\nu}=0 \tag{42}$$

These express that the Coriolis potential field and the velocity field are each orthogonal to the angular velocity field. The tensor of vorticity ω_{ik} defined by

$$\omega_{ik}=\tfrac{1}{2}(C_{i,k}-C_{k,i}) \tag{43}$$

is constant along the vortex filament and in time. This is the relativistic generalization of the classical law of circulation (Lichnerowicz, 1955; Tauber and Weinberg, 1961).

For irrotational motion

$$\operatorname{curl}\mathbf{C}=0 \tag{44}$$

which reduces in view of (40) to

$$C_3 = C(x^3) \tag{45}$$

C being a function of x^3 alone. Hence for motion to be irrotational C_3 must necessarily be independent of x^2. The condition is also sufficient. On the other hand, the rotational motion necessitates the dependence of C_3 on x^2. We note in passing that in the case of irrotational motion all the C_is may be reduced to zero. Hence for irrotational motion one can, without any loss of generality, introduce the coordinates defined by

$$g_{i4} = 0 \tag{46}$$

4. THE CAUCHY PROBLEM

Now we have six field equations (22) together with four consistency conditions (24) to determine six metric tensor components g_{ik}. In default of an exact solution of (22) we venture to obtain solution in powers of time coordinate x^4 and in terms of the Cauchy data on a given 3-space S (Fourès-Bruhat, 1948, 1950, 1952, 1955, 1962; Hadamard, 1932; Lichnerowicz, 1955; Pham Mau Quan, 1953, 1955; Synge, 1960). Without any loss of generality we choose a coordinate system in which the hypersurface S, oriented in space, can be given by $x^4 = 0$. Consequently the normal to the hypersurface S must be oriented in time. As one can obtain the derivatives $g_{ik,j}$ on S directly, it is sufficient to assign as the Cauchy data on S the values of the twelve quantities

$$g_{ik}, \qquad g_{ik,4} \tag{47}$$

chosen subject to the conditions (24). Thus our problem reduces to solve equations (22) subject to the constraints (24).

We confine our investigation to the irrotational motion of a perfect fluid for which one can make

$$C_i = 0 \tag{48}$$

Then by virtue of (2), (4), (7), (29), (30), and (48), equations (22) may be expressed as

$$\begin{aligned} g_{ik,44} = {} & 2\bar{R}_{ik}/\mu^2 - (\dot{\mu}/\mu) g_{ik,4} - \tfrac{1}{2}\xi g_{ik,4} \\ & + g^{mn} g_{im,4} g_{kn,4} + 8\pi(\bar{\rho} - 2p) g_{ik}/\mu^2 \end{aligned} \tag{49}$$

where $\bar{R}_{ik}$ is the intrinsic Ricci tensor of the 3-space $x^4=0$ and

$$\xi = g^{ik} g_{ik,4} \tag{50}$$

Equations (49) explicitly determine the values of $g_{ik,44}$ on S in terms of the Cauchy data (47). It is evident from (49) that the second time derivatives $g_{ik,44}$ do not contain the terms $g_{\nu 4,44}$ which are needed to determine the time evolution of the metric from the Cauchy data on S. Consequently we have a problem of underdetermination in the field equations. As pointed out by Adler et al. (1965) one can get rid of this difficulty by choosing

$$g_{\nu 4,44}=0 \text{ on } x^4=0 \tag{51}$$

In the case under investigation the condition (51) gives in view of (25)

$$\mu=(ax^4+b)^{-1/2}=k+1 \tag{52}$$

where a and b are arbitrary constants.

Thus provided the Cauchy data are chosen subject to the compatibility conditions (24), the solution in the neighborhood of $x^4=0$ may be given by

$$g_{ik}=(g_{ik})_0+x^4(g_{ik,4})_0+\tfrac{1}{2}(x^4)^2(g_{ik,44})_0+\cdots \tag{53}$$

Finally we have to consider the compatibility conditions (24). It is well known that once the compatibility conditions (24) are satisfied on $x^4=0$, they will be automatically satisfied for all time. Equations (24) may thus be interpreted as integrals of the differential system (22). By setting

$$\psi_{ik}=g_{ik,4}, \qquad \text{for } x^4=0 \tag{54}$$

one may write the compatibility conditions (24) as

$$\psi^k_{k,i}-\psi^k_{i\|k}=0 \tag{55}$$

and

$$\bar{R}-\frac{a}{2b^2}\psi^k_k-\frac{1}{4}(\psi^k_k)^2+\frac{3b+4}{4b}\psi^i_j\psi^j_i$$

$$-\frac{b-1}{b}\xi+\frac{16\pi\rho(1+\epsilon)}{b}=0 \tag{56}$$

where $\bar{R}$ is the intrinsic curvature invariant of the hypersurface $x^4=0$, the double bars indicate covariant derivative with respect to Christoffel symbols

$$\Gamma^i_{jk}=g^{im}[jk,m] \tag{57}$$

and ξ is given by

$$\xi=g^{ik}g_{ik,44} \tag{58}$$

Now the four equations (55) and (56) are to be satisfied by twelve quantities g_{ik} and ψ_{ik}. One may obtain a solution corresponding to a linear approximation by setting (Synge, 1966)

$$g_{ik}=\delta_{ik}+\gamma_{ik} \tag{59}$$

and treating γ_{ik} and ψ_{ik} as small. Then the equations (55) and (56) reduce to

$$\psi_{kk,i}-\psi_{ik,k}=0 \tag{60}$$

$$\Delta\gamma_{ii}-\gamma_{ik,ik}+\frac{16\pi\rho(1+\epsilon)/b}{b}=0 \tag{61}$$

where Δ is the Euclidean Laplace operator.

One may obtain the solution of these linearized equations (60) and (61) as

$$\psi_{ik}=0 \tag{62}$$

and

$$\gamma_{ik}=\frac{2\delta_{ik}/b}{b}\int[\rho(1+\epsilon)/r]\,dv, \tag{63}$$

where $dv=dx^1\,dx^2\,dx^3$ and r is the spatial distance (in Euclidean metric) of dv from the point at which γ_{ik} are computed and the integration is over the hypersurface $x^4=0$.

Now the equations (62) imply that the term containing x^4 in the series (53) vanishes and the series contains only even powers of x^4. If one takes

$$k=\text{const},\qquad g_{44}=-1 \tag{64}$$

one may arrive at the solution obtained by Bera and Datta (1974, 1975). Obviously the case (64) leads to time-symmetric solutions (Araki, 1959, Brill, 1959a, 1959b; Fourès-Bruhat, 1955; Misner et al., 1973; Weber and Wheeler, 1957).

In an incoherent fluid or dust cloud defined by

$$p=0 \tag{65}$$

the stream lines are geodesics (Synge, 1966).

Pachner (1968, 1971) has computed the metric tensor components g_{ik} by reducing the system of Einstein partial differential equations to an equivalent system of simultaneous ordinary differential equations in which the numerical integration can be carried out, whereas our solution for g_{ik} corresponds to a linear approximation. The solution thus obtained here is distinct from Pachner's. The solution obtained by Bera and Datta (1974, 1975) is a special case of the solution presented here. Basu et al. (to be published) have, of late, investigated the isentropic motions of a perfect fluid by using comoving coordinates, and a solution corresponding to linear approximation has been obtained for the case

$$\mu=\mu(x^{\nu}) \tag{66}$$

where $\mu(x^{\nu})$ is a function of x^1, x^2, x^3, and x^4. We propose to pursue the study in our next paper, a more general case.

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